An Approximation Algorithm for MINIMUM CONVEX COVER with Logarithmic Performance Guarantee

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Abstract. The problem MINIMUM CONVEX COVER of covering a given polygon with a minimum number of (possibly overlapping) convex polygons is known to be NP-hard, even for polygons without holes [3]. We propose a polynomial-time approximation algorithm for this problem for polygons with or without holes that achieves an approximation ratio of $O(\log n)$, where n is the number of vertices in the input polygon. To obtain this result, we first show that an optimum solution of a restricted version of this problem, where the vertices of the convex polygons may only lie on a certain grid, contains at most three times as many convex polygons as the optimum solution of the unrestricted problem. As a second step, we use dynamic programming to obtain a convex polygon which is maximum with respect to the number of "basic triangles" that are not yet covered by another convex polygon. We obtain a solution that is at most a logarithmic factor off the optimum by iteratively applying our dynamic programming algorithm. Furthermore, we show that MINIMUM CONVEX COVER is APX-hard, i.e., there exists a constant $\delta > 0$ such that no polynomial-time algorithm can achieve an approximation ratio of $1 + \delta$. We obtain this result by analyzing and slightly modifying an already existing reduction [3].

1 Introduction and Problem Definition

The problem MINIMUM CONVEX COVER is the problem of covering a given polygon T with a minimum number of (possibly overlapping) convex polygons that lie in T. This problem belongs to the family of classic art gallery problems; it is known to be NP-hard for input polygons with holes [14] and without holes [3]. The study of approximations for hard art gallery problems has rarely led to good algorithms or good lower bounds; we discuss a few exceptions below. In this paper, we propose the first non-trivial approximation algorithm for MINIMUM CONVEX COVER. Our algorithm works for both, polygons with and without holes. It relies on a strong relationship between the continuous, original problem version and a particular discrete version in which all relevant points are restricted to lie on a kind of grid that we call a quasi-grid. The quasi-grid is the set of intersection points of all lines connecting two vertices of the input polygon. Now, in the RESTRICTED MINIMUM CONVEX COVER problem, the vertices of

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the convex polygons that cover the input polygon may only lie on the quasi-grid. We prove that an optimum solution of the RESTRICTED MINIMUM CONVEX COVER problem needs at most three times the number of convex polygons that the MINIMUM CONVEX COVER solution needs. To find an optimum solution for the RESTRICTED MINIMUM CONVEX COVER problem, we propose a greedy approach: We compute one convex polygon of the solution after the other, and we pick as the next convex polygon one that covers a maximum number of triangles defined on an even finer quasi-grid, where these triangles are not yet covered by previously chosen convex polygons. We propose an algorithm for finding such a maximum convex polygon by means of dynamic programming. To obtain an upper bound on the quality of the solution, we interpret our covering problem on triangles as a special case of the general MINIMUM SET COVER problem that gives as input a base set of elements and a collection of subsets of the base set, and that asks for a smallest number of subsets in the collection whose union contains all elements of the base set. In our special case, each triangle is an element, and each possible convex polygon is a possible subset in the collection, but not all of these subsets are represented explicitly (there could be an exponential number of subsets). This construction translates the logarithmic quality of the approximation from MINIMUM SET COVER to MINIMUM CONVEX COVER [10].

On the negative side, we show that MINIMUM CONVEX COVER is APX-hard, i.e., there exists a constant $\delta > 0$ such that no polynomial-time algorithm can achieve an approximation ratio of $1 + \delta$. This inapproximability result is based on a known problem transformation [3]; we modify this transformation slightly and show that it is gap-preserving (as defined in [1]).

As for previous work, the related problem of partitioning a given polygon into a minimum number of non-overlapping convex polygons is polynomially solvable for input polygons without holes [2]; it is NP-hard for input polygons with holes [12], even if the convex partition must be created by cuts from a given family of (at least three) directions [13]. Other related results for art gallery problems include approximation algorithms with logarithmic approximation ratios for MINIMUM VERTEX GUARD and MINIMUM EDGE GUARD [8], as well as for the problem of covering a polygon with rectangles in any orientation [9]. Furthermore, logarithmic inapproximability results are known for MINIMUM POINT/VERTEX/EDGE GUARD for polygons with holes [5], and APX-hardness results for the same problems for polygons without holes [6]. The related problem RECTANGLE COVER of covering a given orthogonal polygon with a minimum number of rectangles can be approximated with a constant ratio for polygons without holes [7] and with an approximation ratio of $O(\sqrt{\log n})$ for polygons with holes [11]. For additional results see the surveys on art galleries [15,16]. The general idea of using dynamic programming to find maximum convex structures has been used before to solve the problem of finding a maximum (with respect to the number of vertices) empty convex polygon, given a set of vertices in the plane [4], and for the problem of covering a polygon with rectangles in any orientation [9].

This paper is organized as follows: In Sect. 2, we define the quasi-grid and its refinement into triangles. Section 3 contains the proof of the linear relationship between the sizes of the optimum solutions of the unrestricted and restricted

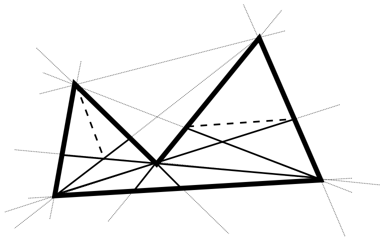

Fig. 1. Construction of first-order basic triangles

convex cover problems. We propose a dynamic programming algorithm to find a maximum convex polygon in Sect. 4, before showing how to iteratively apply this algorithm to find a convex cover in Sect. 5. In Sect. 6, we present the outline of our proof of the APX-hardness of MINIMUM CONVEX COVER. A few concluding remarks can be found in Sect. 7.

2 From the Continuous to the Discrete

Consider simple input polygons with and without holes, where a polygon T is given as an ordered list of vertices in the plane. If T contains holes, each hole is also given as an ordered list of vertices. Let V_T denote the set of vertices (including the vertices of holes, if any) of a given polygon T. While, in the general MINIMUM CONVEX COVER problem, the vertices of the convex polygons that cover the input polygon can be positioned anywhere in the interior or on the boundary of the input polygon, we restrict their positions in an intermediate step: They may only be positioned on a quasi-grid in the RESTRICTED MINIMUM CONVEX COVER problem.

In order to define the RESTRICTED MINIMUM CONVEX COVER problem more precisely, we partition the interior of a polygon T into *convex components* (as proposed in [8] for a different purpose) by drawing a line through each pair of vertices of T. We then triangulate each convex component arbitrarily. We call the triangles thus obtained *first-order basic triangles*. Figure 1 shows in an example the first-order basic triangles of a polygon (thick solid lines) with an arbitrary triangulation (fine solid lines and dashed lines). If a polygon T consists of n vertices, drawing a line through each pair of vertices of T will yield less than $\binom{n}{2} \cdot \binom{n}{2} \in O(n^4)$ intersection points. Let V_T^1 be the set of these intersection points that lie in T (in the interior or on the boundary). Note that $V_T \subseteq V_T^1$. The first-order basic triangles are a triangulation of V_T^1 inside T, therefore the number of first-order basic triangles is also $O(n^4)$. The RESTRICTED MINIMUM CONVEX COVER problem asks for a minimum number of convex polygons, with vertices restricted to V_T^1, that together cover the input polygon T. We call V_T^1 a quasi-grid that is imposed on T. For solving the RESTRICTED MINIMUM CONVEX COVER problem, we make use of a finer quasi-grid: Simply partition T by drawing lines through each pair of points from V_T^1. This yields again convex components, and we triangulate them again arbitrarily. This higher resolution partition yields $O(n^{16})$ intersection points, which define the set V_T^2. We call the resulting triangles *second-order basic triangles*. Obviously, there are $O(n^{16})$ second-order basic triangles. Note that $V_T \subseteq V_T^1 \subseteq V_T^2$.

3 The Optimum Solution of Minimum Convex Cover vs. the Optimum Solution of Restricted Minimum Convex Cover

The quasi-grids V_T^1 and V_T^2 serve the purpose of making a convex cover computationally efficient while at the same time guaranteeing that the cover on the discrete quasi-grid is not much worse than the desired cover in continuous space. The following theorem proves the latter.

Theorem 1. *Let T be an arbitrary simple input polygon with n vertices. Let OPT denote the size of an optimum solution of* Minimum Convex Cover *with input polygon T and let OPT' denote the size of an optimum solution of* Restricted Minimum Convex Cover *with input polygon T. Then:*

$$OPT' \leq 3 \cdot OPT$$

Proof. We proceed as follows: We show how to *expand* a given, arbitrary convex polygon C to another convex polygon C' with $C \subseteq C'$ by iteratively expanding edges. We then replace the vertices in C' by vertices from V_T^1, which results in a (possibly) non-convex polygon C'' with $C' \subseteq C''$. Finally, we describe how to obtain three convex polygons C_1'', C_2'', C_3'' with $C'' = C_1'' \cup C_2'' \cup C_3''$ that only contain vertices from V_T^1. This will complete the proof, since each convex polygon from an optimum solution of Minimum Convex Cover can be replaced by at most 3 convex polygons that are in a solution of Restricted Minimum Convex Cover. Following this outline, let us present the proof details.

Let C be an arbitrary convex polygon inside polygon T. Let the vertices of C be given in in clockwise order. We obtain a series of convex polygons $C^1, C^2, \ldots, C'$ with $C = C^0 \subseteq C^1 \subseteq C^2 \subseteq \ldots \subseteq C'$, where C^{i+1} is obtained from C^i as follows (see Fig. 2):

Let a, b, c, d be consecutive vertices (in clockwise order) in the convex polygon C^i that lies inside polygon T. Let vertices $b, c \notin V_T$, with b and c not on the same edge of T. Then, the edge (b, c) is called *expandable*. If there exists no expandable edge in C^i, then $C' = C^i$, which means we have found the end of the series of convex polygons. If (b, c) is an expandable edge, we *expand* the edge from vertex b to vertex c as follows:

- If b does not lie on the boundary of T, then we let a point p start on b and move on the halfline through a and b away from b until either one of two events happens: p lies on the line through c and d, or the triangle p, c, b touches the boundary of T. Fix p as soon as the first of these events happens. Figure 2 shows a list of all possible cases, where the edges from polygon T are drawn as thick edges: Point p either lies on the intersection point of the lines from a through b and from c through d as in case (a), or there is a vertex v_l on the line segment from p to c as in case (b), or p lies on an edge of T as in case (c).

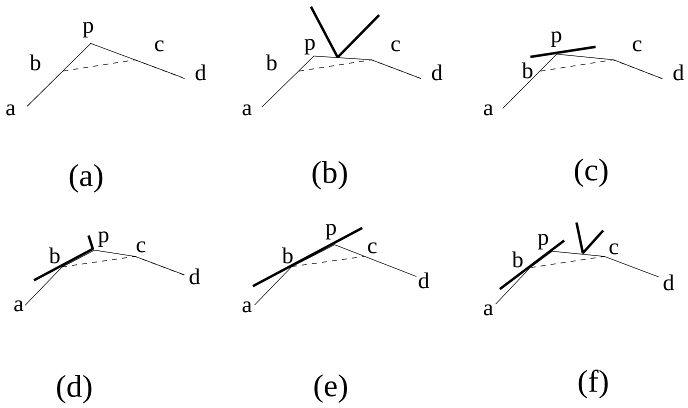

Fig. 2. Expansion of edge (b,c)

– If b lies on the boundary of T, i.e. on some edge of T, say from v_k to v_{k+1}, then let p move as before, except that the direction of the move is now on the way from v_k through b up until v_{k+1} at most (instead of the ray from a through b).

  Figure 2 shows a list of all possible cases: Point p either lies at vertex v_{k+1} as in case (d), or on the intersection point of the lines from b to v_{k+1} and from d through c as in case (e), or there is a vertex v_l on the line segment from p to c as in case (f).

A new convex polygon C_p^i is obtained by simply adding point p as a vertex in the ordered set of vertices of C^i between the two vertices b and c. Furthermore, eliminate all vertices in C_p^i that have collinear neighbors and that are not vertices in V_T.

Note that an edge from two consecutive vertices b and c with $b, c \notin V_T$ can always be expanded in such a way that the triangle b, p, c that is added to the convex polygon is non-degenerate, i.e., has non-zero area, unless b and c both lie on the same edge of polygon T. This follows from the cases (a) - (f) of Fig. 2.

Now, let $C^{i+1} = C_p^i$, if either a new vertex of T has been added to C_p^i in the expansion of the edge, which is true in cases (b), (d), and (f), or the number of vertices of C_p^i that are not vertices of T has decreased, which is true in case (a). If p is as in case (c), we expand the edge (p, c), which will result in either case (d), (e), or (f). Note that in cases (d) and (f), we have found C^{i+1}. If p is as in case (e), we expand the edge (p, d), which will result in either case (d), (e), or (f). If it is case (e) again, we repeat the procedure by expanding the edge from p and the successor (clockwise) of d. This needs to be done at most $|C^i|$ times, since the procedure will definitely stop once it gets to vertex a. Therefore, we obtain C^{i+1} from C^i in a finite number of steps. Let τ_i denote the number of vertices in C^i that are also vertices in T and let $\hat{\tau}_i$ be the number of vertices in C^i that are not vertices in T. Now note that $\phi(i) = \hat{\tau}_i - 2\tau_i + 2n$ is a function that bounds the number of remaining steps, i.e., it strictly decreases with every increase in

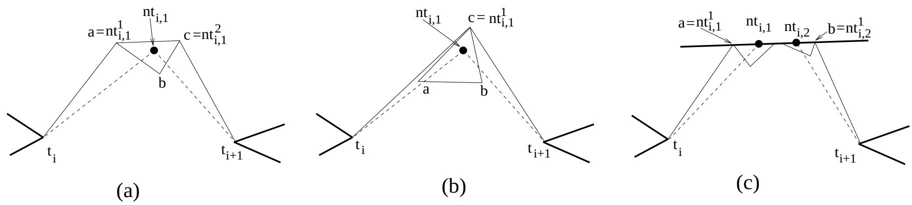

Fig. 3. Replacing non-T-vertices

i and cannot become negative. The existence of this bounding function implies the finiteness of the series $C^1, C^2, \ldots, C'$ of convex polygons.

By definition, there are no expandable edges left in C'. Call a vertex of C' a T-vertex, if it is a vertex in T. From the definition of expandable edges, it is clear that there can be at most two non-T-vertices between any two consecutive T-vertices in C', and if there are two non-T-vertices between two consecutive T-vertices, they must both lie on the same edge in T. Let the T-vertices in C' be $t_1, \ldots, t_l$ in clockwise order, and let the non-T-vertices between t_i and t_{i+1} be $nt_{i,1}$ and $nt_{i,2}$ if they exist. We now replace each non-T-vertex $nt_{i,j}$ in C' by one or two vertices $nt^1_{i,j}$ and $nt^2_{i,j}$ that are both elements of V^1_T. This will transform the convex polygon C' into a non-convex polygon C'' (we will show later how C'' can be covered by at most three convex polygons C''_1, C''_2, C''_3).

To this end, let a, b, c be the first-order basic triangle in which non-T-vertex $nt_{i,j}$ lies, as illustrated in Fig. 3. Points a, b, c are all visible from both vertices t_i and t_{i+1}. To see this, assume by contradiction that the view from, say, t_i to a is blocked by an edge e of T. Since $nt_{i,j}$ must see t_i, the edge e must contain a vertex e' in the triangle $t_i, a, nt_{i,j}$, but then a cannot be a vertex of the first-order basic triangle in which $nt_{i,j}$ lies, since the line from vertex t_i through vertex e' would cut through the first-order basic triangle, an impossibility. Now, let d_i be the intersection point of the line from t_{i-1} through t_i and the line from t_{i+1} through t_{i+2}. With similar arguments, the triangle t_i, d_i, t_{i+1} completely contains triangle a, b, c.

Assume that only one non-T-vertex $nt_{i,1}$ exists between t_i and t_{i+1}. If the triangle formed by t_i, t_{i+1} and a completely contains the triangle $t_i, nt_{i,1}, t_{i+1}$, we let $nt^1_{i,1} = a$, likewise for b and c (see Fig. 3 (b)). Otherwise, we let $(nt^1_{i,1}, nt^2_{i,1})$ be $(a, b), (a, c)$, or (b, c) as in Fig. 3 (a), such that the polygon $t_i, nt^1_{i,1}, nt^2_{i,1}, t_{i+1}$ is convex and completely contains the triangle $t_i, nt_{i,1}, t_{i+1}$. This is always possible by the definition of points a, b, c.

Now, assume that two non-T-vertices $nt_{i,1}$ and $nt_{i,2}$ exist between t_i and t_{i+1}. From the definition of C', we know that $nt_{i,1}$ and $nt_{i,2}$ must lie on the same edge e of T. Therefore, the basic triangle in which $nt_{i,1}$ lies must contain a vertex a either at $nt_{i,1}$ or preceeding $nt_{i,1}$ on edge e along T in clockwise order.

Let $nt^1_{i,1} = a$. The basic triangle in which $nt_{i,2}$ lies must contain a vertex b either at $nt_{i,2}$ or succeeding $nt_{i,2}$ on edge e. Let $nt^1_{i,2} = b$. See Fig. 3 (c). Note that the convex polygon $t_i, nt^1_{i,1}, nt^1_{i,2}, t_{i+1}$ completely contains the polygon $t_i, nt_{i,1}, nt_{i,2}, t_{i+1}$.

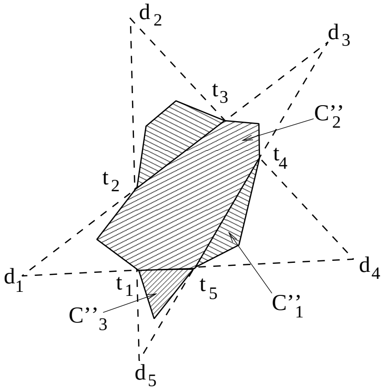

Fig. 4. Covering C'' with three convex polygons

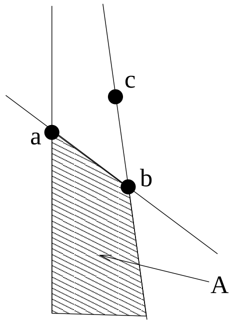

Fig. 5. Dynamic Programming

After applying this change to all non-T-vertices in C', we obtain a (possibly) non-convex polygon C''. First, assume that C'' contains an odd number f of T-vertices. We let C''_1 be the polygon defined by vertices $t_i, nt^k_{i,j}$ and t_{i+1} for all j, k and for all odd i, but $i \neq f$. By construction, C''_1 is convex. Let C''_2 be the polygon defined by vertices $t_i, nt^k_{i,j}$ and t_{i+1} for all j, k and for all even i. Finally, let C''_3 be the polygon defined by vertices $t_f, nt^k_{f,j}$ and t_1 for all j, k. Figure 4 shows an example. Obviously, C''_1, C''_2, and C''_3 are convex and together cover all of C''. Second, assume that C'' contains an even number of T-vertices, and cover it with only two convex polygons using the same concept. This completes the proof.

4 Finding Maximum Convex Polygons

Assume that each second-order basic triangle from a polygon T is assigned a weight value of either 1 or 0. In this section, we present an algorithm using dynamic programming that computes the convex polygon M in a polygon T that contains a maximum number of second-order basic triangles with weight 1 and that only has vertices from V^1_T. For simplicity, we call such a polygon a *maximum convex polygon*. The weight of a polygon M is defined as the sum of the weights of the second-order basic triangles in the polygon and is denoted by $|M|$. We will later use the algorithm described below to iteratively compute a maximum convex polygon with respect to the triangles that are not yet covered, to eventually obtain a convex cover for T.

Let $a, b, c \in V^1_T$. Let $P_{a,b,c}$ denote the maximum convex polygon that:

– contains only vertices from V^1_T, and
– contains vertices a, b, c in counterclockwise order, and
– has a as its left-most vertex, and
– contains additional vertices only between vertices a and b.

Given three vertices $a, b, c \in V^1_T$, let A be the (possibly infinite) area of points that are:

– to the right of vertex a, and
– to the left of the line oriented from b through a, and
– to the left of the line oriented from b through c.

For an illustration, see Fig. 5. Let $P'_{a,b,c} = \max_{d \in V_T^1 \cap A} P_{a,d,b} \cup \Delta a,b,c$, where max is defined as follows (to simplify notation):

$$\max\{P_1, P_2\} = \begin{cases} P_1 \text{ if } |P_1| \geq |P_2| \\ P_2 \text{ otherwise} \end{cases} .$$

Lemma 1. $P_{a,b,c} = P'_{a,b,c}$, *if the triangle* a,b,c *is completely contained in the polygon* T.

Proof. Consider $P_{a,b,c}$, which is maximum by definition. $P_{a,b,c}$ must contain additional vertices between a and b (otherwise the lemma is trivially true). Let d' be the predecessor of b in the counterclockwise order of $P_{a,b,c}$. Vertex d' must lie in area A as defined above, otherwise the polygon a,d',b,c would either be non-convex, not have a as its left-most vertex, or not be in the required counterclockwise order. Now consider $P'' = P_{a,b,c} - \Delta a,b,c$. From the definition of area A it is clear the P'' can only contain vertices that lie in A. Now $P_{a,d',b}$ is maximum by definition, and it is considered when computing $P'_{a,b,c}$.

Let M be a maximum convex polygon for a polygon T with weights assigned to the second-order basic triangles. Let a be the left-most vertex of M, let c be the predecessor of a in M in counter clockwise order, and let b be the predecessor of c. Then $|P_{a,b,c}| = |M|$ by definition.

We will now use Lemma 1 to construct an algorithm, which takes as input a polygon T and an assignment of weight 0 or 1 to each second-order basic triangle of T and computes the maximum convex polygon. To this end, we fix vertex $a \in V_T^1$. Let a' be a point with the same x-coordinate and smaller y-coordinate than a. Now, order all other vertices $b \in V_T^1$ to the right of a according to the angle formed by b,a,a'. Let the resulting ordered set be B and let B' be the empty set. Take the smallest element b from B, remove it from B and add it to set B', then for all $c \in V_T^1 \backslash B'$ and to the right of a, compute weight $|\Delta a,b,c|$ of the triangle a,b,c and compute $P_{a,b,c}$ according to Lemma 1. Compute weight $|P_{a,b,c}|$ by adding $|\Delta a,b,c|$ to $|P_{a,d,b}|$, where d is the maximizing argument. Note that the computation of $P_{a,b,c}$ according to Lemma 1 is always possible, since all possible vertices d in $P_{a,d,b}$ lie to the left of the line from b to a (see also definition of area A) and have therefore smaller angles d,a,a' than b,a,a', and have therefore already been computed. The algorithm is executed for every $a \in V_T^1$, and the maximum convex polygon found is returned.

Note that $|T| = n$, $|V_T^1| = O(n^4)$, and $|V_T^2| = O(n^{16})$. Ordering $O(n^4)$ vertices takes $O(n^4 \log n)$ time. Computing the weight of a triangle takes $O(n^{16})$ time. Computing $P_{a,b,c}$ takes $O(n^4)$ time. We have to compute the weight of $O(n^8)$ triangles, which gives a total time of $O(n^{24})$. Finally, we have to execute our algorithm for each $a \in V_T^1$, which gives a total running time of $O(n^{28})$. Space requirements are $O(n^{12})$ by using pointers.

5 An Approximation Algorithm for MINIMUM CONVEX COVER

Given a polygon T, we obtain a convex cover by iteratively applying the algorithm for computing a maximum convex polygon from Sect. 4. It works as follows for an input polygon T.

1. Let all second-order basic triangles have weight 1. Let $S = \emptyset$.
2. Find the maximum convex polygon M of polygon T using the algorithm from Sect. 4, and add M to the solution S. Decrease the weight of all second-order basic triangles that are contained in M to 0.[1]
3. Repeat step 2 until there are no second-order basic units with weight 1 left. Return S.

To obtain a performance guarantee for this algorithm, consider the MINIMUM SET COVER instance I, which has all second-order basic triangles as elements and where the second-order basic triangles with weight 1 of each convex polygon in T, which only contains vertices from V_T^1, form a set in I. The greedy heuristic for MINIMUM SET COVER achieves an approximation ratio of $1 + \ln n'$, where n' is the number of elements in I [10] and it works in exactly the same way as our algorithm. However, we do not have to (and could not afford to) compute all the sets of the MINIMUM SET COVER instance I (which would be a number exponential in n'): It suffices to always compute a set, which contains a maximum number of elements not yet covered by the solution thus far. This is achieved by reducing the weights of the second-order basic triangles already in the solution to 0; i.e. a convex polygon with maximum weight is such a set.

Note that $n' = O(n^{16})$. Therefore, our algorithm achieves an approximation ratio of $O(\log n)$ for RESTRICTED MINIMUM CONVEX COVER on input polygon T. Because of Theorem 1, we know that the solution found for RESTRICTED MINIMUM CONVEX COVER is also a solution for the unrestricted MINIMUM CONVEX COVER that is at most a factor of $O(\log n)$ off the optimum solution.

As for the running time of this algorithm, observe that the algorithm adds to the solution in each round a convex polygon with non-zero weight. Therefore, there can be at most $O(n^{16})$ rounds, which yields a total running time of $O(n^{44})$. This completes the proof of our main theorem:

Theorem 2. MINIMUM CONVEX COVER *for input polygons with or without holes can be approximated by a polynomial time algorithm with an approximation ratio of* $O(\log n)$*, where* n *is the number of polygon vertices.*

6 *APX*-Hardness of MINIMUM CONVEX COVER

The upper bound of $O(\log n)$ on the approximation ratio for MINIMUM CONVEX COVER is not tight: We will now prove that there is a constant lower bound on the approximation ratio, and hence a gap remains. More precisely, we prove

[1] Note that by the definition of second-order basic triangles, a second-order basic triangle is either completely contained in M or completely outside M.

Minimum Convex Cover to be APX-hard. Our proof of the APX-hardness of Minimum Convex Cover for input polygons with or without holes uses the construction that is used to prove the NP-hardness of this problem for input polygons without holes[2] [3]. However, we reduce the problem Maximum 5-Occurrence-3-Sat rather than SAT (as done in the original reduction [3]) to Minimum Convex Cover, and we design the reduction to be gap-preserving [1]. Maximum 5-Occurrence-3-Sat is the variant of SAT, where each variable may appear at most 5 times in clauses and each clause contains at most 3 literals. Maximum 5-Occurrence-3-Sat is APX-complete [1].

Theorem 3. *Let I be an instance of* Maximum 5-Occurrence-3-Sat *consisting of n variables, m clauses with a total of l literals, and let I' be the corresponding instance of* Minimum Convex Cover. *Let OPT be the maximum number of satisfied clauses of I by any assignment of the variables. Let OPT' be the minimum number of convex polygons needed to cover the polygon of I'. Then:*

$$OPT = m \Longrightarrow OPT' = 5l + n + 1$$
$$OPT \leq (1 - 15\epsilon)m \Longrightarrow OPT' \geq 5l + n + 1 + \epsilon n$$

Proof. Theorem 3 is proved by showing how to transform the convex polygons of a solution of the Minimum Convex Cover I' in such a way that their total number does not increase and in such a way that a truth assignment of the variables can be “inferred” from the convex polygons that satisfies the desired number of clauses. The proof employs concepts similar to those used in [6]; we do not include details, due to space limitation.

In the promise problem of Maximum 5-Occurrence-3-Sat as described above, we are promised that either all clauses are satisfiable or at most a fraction of $1 - 15\epsilon$ of the clauses is satisfiable, and we are to find out, which of the two possibilities is true. This problem is NP-hard for small enough values of $\epsilon > 0$ [1]. Therefore, Theorem 3 implies that the promise problem for Minimum Convex Cover, where we are promised that the minimum solution contains either $5l+n+1$ convex polygons or $5l+n+1+\epsilon n$ convex polygons, is NP-hard as well, for small enough values of $\epsilon > 0$. Therefore, Minimum Convex Cover cannot be approximated with a ratio of: $\frac{5l+n+1+\epsilon n}{5l+n+1} \geq 1 + \frac{\epsilon n}{25n+n+1} \geq 1 + \frac{\epsilon}{27}$, where we have used that $l \leq 5n$ and $n \geq 1$. This establishes the following:

Theorem 4. Minimum Convex Cover *on input polygons with or without holes is APX-hard.*

7 Conclusion

We have proposed a polynomial time approximation algorithm for Minimum Convex Cover that achieves an approximation ratio that is logarithmic in the

2 APX-hardness for Minimum Convex Cover for input polygons without holes implies APX-hardness for the same problem for input polygons with holes.

number of vertices of the input polygon. This has been achieved by showing that there is a discretized version of the problem using no more than three times the number of cover polygons. The discretization may shed some light on the long-standing open question of whether the decision version of the MINIMUM CONVEX COVER problem is in NP [15]. We know now that convex polygons of optimum solutions only contain a polynomial number of vertices and that a considerable fraction of these vertices are actually vertices from the input polygon. Apart from the discretization, our algorithm applies a MINIMUM SET COVER approximation algorithm to a MINIMUM SET COVER instance with an exponential number of sets that are represented only implicitly, through the geometry. We propose an algorithm that picks a best of the implicitly represented sets with a dynamic programming approach, and hence runs in polynomial time. This technique may prove to be of interest for other problems as well. Moreover, by showing APX-hardness, we have eliminated the possibility of the existence of a polynomial-time approximation scheme for this problem. However, polynomial time algorithms could still achieve constant approximation ratios. Whether our algorithm is the best asymptotically possible, is therefore an open problem. Furthermore, our algorithm has a rather excessive running time of $O(n^{44})$, and it is by no means clear whether this can be improved substantially.

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References

1. S. Arora and C. Lund; *Hardness of Approximations*; in: Approximation Algorithms for NP-Hard Problems (ed. Dorit Hochbaum), PWS Publishing Company, pp. 399-446, 1996.
2. B. Chazelle, D.P. Dobkin; *Optimal Convex Decompositions*; Computational Geometry, pp. 63-133, Elsevier Science B. V., 1985.
3. J. C. Culberson and R. A. Reckhow; *Covering Polygons is Hard*; Journal of Algorithms, Vol. 17, No. 1, pp. 2 – 44, July 1994.
4. D. P. Dobkin, H. Edelsbrunner, and M. H. Overmars; *Searching for Empty Convex Polygons*; Algorithmica, 5, pp. 561 – 571, 1990.
5. S. Eidenbenz, C. Stamm, and P. Widmayer; *Inapproximability of some Art Gallery Problems*; Proc. 10th Canadian Conf. Computational Geometry, pp. 64-65, 1998.
6. S. Eidenbenz; *Inapproximability Results for Guarding Polygons without Holes*; Lecture Notes in Computer Science, Vol. 1533 (ISAAC'98), p. 427-436, 1998.
7. D.S. Franzblau; *Performance Guarantees on a Sweep Line Heuristic for Covering Rectilinear Polygons with Rectangles*; SIAM J. Discrete Math., Vol 2,3, pp. 307 – 321, 1989.
8. S. Ghosh; *Approximation Algorithms for Art Gallery Problems*; Proc. of the Canadian Information Processing Society Congress, 1987.
9. J. Gudmundsson, C. Levcopoulos; *Close Approximations of Minimum Rectangular Coverings*; J. Comb. Optimization, Vol. 4, No. 4, pp. 437 – 452, 1999.
10. D. Hochbaum; *Approximating Covering and Packing Problems: Set Cover, Vertex Cover, Independent Set, and Related Problems*; in: Approximation Algorithms for NP-Hard Problems (ed. Dorit Hochbaum), PWS Publishing Company, pp. 94-143, 1996.

11. V.S.A. Kumar, H. Ramesh; *Covering Rectilinear Polygons with Axis-parallel Rectangles*; Proc. STOC'99, pp. 445 – 454, 1999.
12. A. Lingas; *The Power of Non-Rectilinear Holes*; Proc. 9^{th} Colloquium on Automata, Languages, and Programming, pp. 369 – 383, 1982.
13. A. Lingas, V. Soltan; *Minimum Convex Partition of a Polygon with Holes by Cuts in Given Directions*; Theory of Comuting Systems, Vol. 31, pp. 507 – 538, 1998.
14. J. O'Rourke and K. J. Supowit; *Some NP-hard Polygon Decomposition Problems*; IEEE Transactions on Information Theory, Vol IT-29, No. 2, 1983.
15. T. Shermer; *Recent results in Art Galleries*; Proc. of the IEEE, 1992.
16. J. Urrutia; *Art Gallery and Illumination Problems*; in Handbook on Computational Geometry, ed. J.-R. Sack and J. Urrutia, Elsevier, Chapter 22, pp. 973 – 1027, 2000.